



# No-arbitrage condition and existence of equilibrium with dividends

Cuong Le Van, Nguyen Ba Minh

## ► To cite this version:

Cuong Le Van, Nguyen Ba Minh. No-arbitrage condition and existence of equilibrium with dividends. *Journal of Mathematical Economics*, 2007, 43 (2), pp.135-152. 10.1016/j.jmateco.2006.05.010 . halshs-00101177

**HAL Id: halshs-00101177**

**<https://shs.hal.science/halshs-00101177>**

Submitted on 26 Sep 2006

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# No-Arbitrage Condition and Existence of Equilibrium with Dividends\*

Cuong Le Van<sup>†</sup>, Nguyen Ba Minh<sup>‡</sup>

June 9, 2006

## Abstract

In this paper we first give an elementary proof of existence of equilibrium with dividends in an economy with possibly satiated consumers. We then introduce a no-arbitrage condition and show that it is equivalent to the existence of equilibrium with dividends.

*Journal of economic literature* classification numbers: C62, D50.

## 1 Introduction

In the Arrow-Debreu model (1954), the authors impose a nonsatiation assumption which states that for every consumer, whatever the commodity bundle may be, there exists another consumption bundle she/he strictly prefers. It is well-known, that in presence of satiation, a Walras equilibrium may not exist since for every price, there could be a consumer who maximizes her/his preference in the interior of her/his budget set. In presence of financial assets, satiation is rather a rule than an exception. Both the mean-variance CAPM and the expected-utility model with negative returns exhibit satiation (see e.g. Nielsen (1989), Dana, Le Van and Magnien (1997), Section 5).

---

\*The authors are grateful to an anonymous referee for her/his observations, criticisms and suggestions

<sup>†</sup>Corresponding author, Centre d'Economie de la Sorbonne, University Paris 1 Pantheon-Sorbonne, CNRS, levan@univ-paris1.fr

<sup>‡</sup>Hanoi University of Commerce, baminhdhtm@hotmail.com

The absence of the nonsatiation condition with fixed prices was studied by Drèze and Muller (1980) by introducing the notion of coupons equilibrium, Aumann and Drèze (1986) with the concept of dividends, Mas-Colell (1992) who used the term of slack equilibrium. In Debreu (1959, *Theory of Value*), the notion of an *equilibrium relative to the price system* can be viewed as an equilibrium with possibly negative dividends. We can cite other authors who worked on nonsatiation: e.g. Makarov (1981), Kajii (1996), Florig and Yildiz (2002), Konovalov (2005), and for a continuum of consumers, Cornet, Topuzu and Yildiz (2003).

In this paper we first give an easy proof of existence of equilibria with dividends. For Aumann and Drèze, a dividend is a "cash allowance added to the budget by each trader. Its function is to distribute among the non-satiated agents the surplus created by the failure of the satiated agents to use their entire budget". Here, we introduce an additional good (e.g. financial asset, or paper money) that the satiated agents will want to have in order to fill up their budget sets. For that, they will buy this additional good from the nonsatiated agents. More precisely, we will introduce an intermediary economy by adding another good that any agent would like to have if she/he meets satiation. In this economy, the nonsatiation condition is satisfied. There thus exists a Walras equilibrium. We show that this equilibrium actually corresponds to an equilibrium with dividends for the initial economy. It is interesting to notice that we show that, at this equilibrium, the satiated agents will buy the additional good from the nonsatiated agents and if an agent is not satiated then the value of the additional good will be zero for that agent. It is important to note that the idea to introduce an additional good is not new when one considers the *equilibrium with paper money* of Kajii (1996). What is new in this paper is the mechanism of exchange: it is defined clearly with well-defined partial extended preferences that the satiated consumers who meet satiation points will buy additional good from the consumers who do not meet satiation.

Second, we allow our model to have financial assets. If we assume that the production sets satisfy in particular the inaction and irreversibility conditions (see Debreu, 1959) and the utility functions satisfy the No-Half Line Condition (see e.g. Werner, 1986, Page and Wooders 1996, Dana, Le Van and Magnien, 1999, Allouch, Le Van, Page, 2002), then there exists an equilibrium with dividends iff there exists a no-arbitrage price. Usually, no-arbitrage conditions are introduced in an exchange economy with financial markets. Here, we introduce a no-arbitrage condition in an economy with production. We

think of two-period models where firms produce consumption goods using capital goods and the consumers buy, in the first period, consumption goods and assets. An opportunity of arbitrage is a system of prices of commodities (consumption goods or assets) for which, either at least one consumer, without cost, can increase without bound her/his consumption, or one firm produces more and more because her/his profit increases without bound.

The paper is organized as follows. The model is presented in Section 2. The main result is given in Section 3. In Section 4, we introduce the no-arbitrage price condition and prove that existence of equilibrium is equivalent to existence of no-arbitrage prices. In Section 5, Appendix 1 gives a proof of Theorem 2 of Section 3. In Section 6, Appendix 2 presents an example of economies with production where the no-arbitrage condition is satisfied.

## 2 The Model

We consider an economy having  $l$  goods,  $J$  producers, and  $I$  consumers. We suppose that the numbers of the producers and the consumers are finite. For each  $i \in I$ , let  $X_i \subset \mathbb{R}^l$  denote the set of consumption goods, let  $u_i : X_i \rightarrow \mathbb{R}$  denote the utility and let  $e_i \in \mathbb{R}^l$  be the initial endowment. Furthermore for each  $j \in J$ , let  $Y_j \subset \mathbb{R}^l$  denote the producing set of the producer  $j$ .

Let  $\theta_{ij}$  be the ratio of the profit that consumer  $i$  can get from the producer  $j$ . We suppose that  $0 \leq \theta_{ij} \leq 1$ ,  $\sum_{i \in I} \theta_{ij} = 1$ . Let  $p \in \mathbb{R}^l$  denote the price of the goods.

In the sequel we will denote this economy by

$$\mathcal{E} = \{(X_i, u_i, e_i)_{i \in I}, (Y_j)_{j \in J}, (\theta_{ij})_{i \in I, j \in J}\}.$$

### 2.1 Preliminaries

We recall that a function  $u_i$  is said to be *quasiconcave* if its level-set

$$L^\alpha = \{x_i \in X_i : u_i(x_i) \geq \alpha\}$$

is convex for each  $\alpha \in \mathbb{R}$ .

The function  $u_i$  is *strictly quasiconcave* if and only if  $x_i, x'_i \in X_i$ ,  $u_i(x'_i) > u_i(x_i)$  and  $\lambda \in [0, 1)$ , then

$$u_i(\lambda x_i + (1 - \lambda)x'_i) > u_i(x_i).$$

It means that

$$u_i(\lambda x_i + (1 - \lambda)x'_i) > \min(u_i(x_i), u_i(x'_i)).$$

The function  $u_i$  is *upper semicontinuous* if and only if  $L^\alpha$  is closed for each  $\alpha$ .

Let  $S_i$  denote the set of satiation points of  $u_i$ . Then

$$S_i = \{x'_i \in X_i : u_i(x'_i) \geq u_i(x_i), \text{ for any } x_i \in X_i\}.$$

By this definition, the function  $u_i$  has no satiation point if for all  $x_i \in X_i$  there exists  $x'_i \in X_i$  such that  $u_i(x'_i) > u_i(x_i)$ . It is easy to check that  $S_i$  is convex and closed.

## 2.2 Definition 1

A *Walras equilibrium* of  $\mathcal{E}$  is a list  $((x_i^*)_{i \in I}, (y_j^*)_{j \in J}, p^*) \in (\mathbb{R}^l)^{|I|} \times (\mathbb{R}^l)^{|J|} \times (\mathbb{R}^l \setminus \{0\})$  which satisfies

- (a)  $\sum_{i \in I} x_i^* = \sum_{i \in I} e_i + \sum_{j \in J} y_j^*$  (Market clearing);
- (b) for each  $i$  one has

$$p^* \cdot x_i^* = p^* \cdot e_i + \sum_{j \in J} \theta_{ij} \cdot \sup p^* \cdot Y_j$$

(budget constraint), and for each  $x_i \in X_i$ , with  $u_i(x_i) > u_i(x_i^*)$ , it holds

$$p^* \cdot x_i > p^* \cdot e_i + \sum_{j \in J} \theta_{ij} \cdot \sup p^* \cdot Y_j.$$

- (c) For each  $j \in J$ ,  $y_j^* \in Y_j$  and  $p^* \cdot y_j^* = \sup p^* \cdot Y_j$ , where  $\sup p \cdot Y_j = \sup_{y_j \in Y_j} p \cdot y_j$ .

A *Walras quasi-equilibrium* is a list  $((x_i^*)_{i \in I}, (y_j^*)_{j \in J}, p^*) \in (\mathbb{R}^l)^{|I|} \times (\mathbb{R}^l)^{|J|} \times (\mathbb{R}^l \setminus \{0\})$  which satisfies (a), (c), and (b) with the following change:

$$u_i(x_i) > u_i(x_i^*) \Rightarrow p^* \cdot x_i \geq p^* \cdot e_i + \sum_{j \in J} \theta_{ij} \cdot \sup p^* \cdot Y_j.$$

### 2.3 Definition 2

An *equilibrium with dividends*  $(d_i^*)_{i \in I} \in \mathbb{R}_+^{|I|}$  of  $\mathcal{E}$  is a list  $((x_i^*)_{i \in I}, (y_j^*)_{j \in J}, p^*) \in (\mathbb{R}^l)^{|I|} \times (\mathbb{R}^l)^{|J|} \times \mathbb{R}^l$  which satisfies:

- (a)  $\sum_{i \in I} x_i^* = \sum_{i \in I} e_i + \sum_{j \in J} y_j^*$  (Market clearing);
- (b) for each  $i$  one has

$$p^* \cdot x_i^* \leq p^* \cdot e_i + \sum_{j \in J} \theta_{ij} \cdot \sup p^* \cdot Y_j + d_i^*$$

(budget constraint), and for each  $x_i \in X_i$ , with  $u_i(x_i) > u_i(x_i^*)$ , it holds

$$p^* \cdot x_i > p^* \cdot e_i + \sum_{j \in J} \theta_{ij} \cdot \sup p^* \cdot Y_j + d_i^*$$

- (c) For each  $j \in J$ ,  $y_j^* \in Y_j$  and  $p^* \cdot y_j^* = \sup p^* \cdot Y_j$ , where  $\sup p \cdot Y_j = \sup_{y_j \in Y_j} p \cdot y_j$ .

### 2.4 Definition 3

A *feasible allocation* is the list  $((x_i)_{i \in I}, (y_j)_{j \in J}) \in \prod_{i \in I} X_i \times \prod_{j \in J} Y_j$  which satisfies  $\sum_{i \in I} x_i = \sum_{i \in I} e_i + \sum_{j \in J} y_j$ . We denote by  $A$  the set of feasible allocations and by  $A_i$  the projection of  $A$  on the  $i^{th}$  component.

The main purpose of this paper is to give an easy proof of existence of equilibrium with dividends of economy  $\mathcal{E}$  when satiation points occur in the preferences of the consumers. Our idea is to introduce an intermediary economy with an additional good (think of financial asset or money paper) that the consumers want to possess when they meet satiation. In this new economy, there is no satiation point. Hence, an equilibrium exists under appropriate assumptions. We show that this equilibrium is an equilibrium with dividends for the initial economy. It is worth to point out that at this equilibrium point, the consumers who meet satiation points will buy the additional good from the consumers who do not meet satiation.

### 2.5 The Assumptions

We now list our assumptions.

- (H<sub>1</sub>) For each  $i \in I$ , the set  $X_i$  is nonempty closed convex;

- (H<sub>2</sub>) For each  $i \in I$ , the function  $u_i$  is strictly quasiconcave and upper semicontinuous;
- (H<sub>3</sub>) For each  $j \in J$ , the set  $Y_j$  is nonempty closed convex and  $Y = \sum_{j \in J} Y_j$  is closed.
- (H<sub>4</sub>) The feasible set  $A$  is compact.
- (H<sub>5</sub>) For every  $i$ ,  $e_i \in \text{int}(X_i - \sum_{j \in J} \theta_{ij} Y_j)$ . Moreover, for every  $i \in I$ ,  $x_i \in A_i$  the set  $\{x'_i : u_i(x'_i) > u_i(x_i)\}$  is relatively open in  $X_i$ .

**Remark 1** (1) Assumptions (H<sub>1</sub>), (H<sub>2</sub>) are standard.

(2) Assumption (H<sub>3</sub>) can be relaxed as follows: for each  $j \in J$ , the set  $Y_j$  is nonempty and the total production set  $Y = \sum_j Y_j$  is closed and convex (see Remark 5 (1) below).

(3) Assumption (H<sub>4</sub>) is satisfied when the consumption sets are the positive orthant  $\mathbb{R}_+^l$ , the production sets satisfy  $0 \in Y_j, \forall j$ , the total production set satisfies  $Y \cap (-Y) = \{0\}$  (irreversibility) and  $Y \cap \mathbb{R}_+^l = \{0\}$  (one cannot produce without using input). It is also satisfied in a financial exchange economy with strictly concave utility functions and a no-arbitrage condition (see e.g. Page (1987) or Page and Wooders (1996)). We give in Appendix 2 two examples of economies with production and assets where the no-arbitrage condition is satisfied.

(4) Assumption (H<sub>5</sub>) ensures that any quasi-equilibrium is actually an equilibrium.

### 3 The Results

We first give an existence of Walras equilibrium theorem when there exists no satiation.

**Theorem 2** Assume (H<sub>1</sub>) – (H<sub>4</sub>) and

(i)

$$\forall i, e_i \in (X_i - \sum_{j \in J} \theta_{ij} Y_j)$$

$$\forall i, \forall x_i \in X_i, \exists x'_i \in X_i \text{ such that } u_i(x'_i) > u_i(x_i).$$

then there exists a quasi-equilibrium.

(ii) If we add  $H_5$  and

$$\forall i, \forall x_i \in X_i, \exists x'_i \in X_i \text{ such that } u_i(x'_i) > u_i(x_i),$$

then there exists an equilibrium.

**Proof.** We adapt the proof given in Dana, Le Van and Magnien (1999) for an exchange economy. A detailed proof is given in Appendix 1. ■

We now come to our main result which is a corollary of the previous theorem.

**Theorem 3** Assume  $(H_1) - (H_5)$ . Then there exists an equilibrium with dividends.

**Proof.** Let us introduce the intermediary economy

$$\widehat{\mathcal{E}} = \left\{ (\widehat{X}_i, \widehat{u}_i, \widehat{e}_i)_{i \in I}, (\widehat{Y}_j)_{j \in J}, (\theta_{ij})_{i \in I, j \in J} \right\}$$

where:  $\widehat{X}_i = X_i \times \mathbb{R}_+$ ,  $\widehat{e}_i = (e_i, \delta_i)$  with  $\delta_i > 0$  for any  $i \in I$  and  $\widehat{Y}_j = (Y_j, 0)$  for any  $j \in J$ , and the utilities  $\widehat{u}_i$  are defined as follows (recall that  $S_i$  is the set of satiation points for agent  $i$ ): let  $\mu > 0$ ,  $M_i = \max \{u_i(x) : x \in X_i\}$ .

- If  $x_i \notin S_i$ , then  $\widehat{u}_i(x_i, d_i) = u_i(x_i)$  for any  $d_i \geq 0$ .

- If  $x_i \in S_i$ , then  $\widehat{u}_i(x_i, d_i) = u_i(x_i) + \mu d_i = M_i + \mu d_i$  for any  $d_i \geq 0$ .

We will check that Assumption  $(H_2)$  is satisfied for every  $\widehat{u}_i$ .

To prove that  $\widehat{u}_i$  is quasi-concave and upper semi-continuous, it suffices to prove that the set  $\widehat{L}_i^\alpha = \{(x_i, d_i) \in X_i \times \mathbb{R}_+ : \widehat{u}_i(x_i, d_i) \geq \alpha\}$  is closed and convex for every  $\alpha$ . We have two cases:

**Case 1:**  $\alpha < M_i$ . We claim that  $\widehat{L}_i^\alpha = L_i^\alpha \times \mathbb{R}_+$ . Indeed, let  $(x_i, d_i) \in \widehat{L}_i^\alpha$ . It follows  $\widehat{u}_i(x_i, d_i) \geq \alpha$  and there are two possibilities for  $x_i$ :

+ If  $x_i \notin S_i$ , then  $\widehat{u}_i(x_i, d_i) = u_i(x_i)$ . It implies  $u_i(x_i) \geq \alpha$  or  $x_i \in L_i^\alpha$  and hence  $(x_i, d_i) \in L_i^\alpha \times \mathbb{R}_+$ .

+ If  $x_i \in S_i$ , then  $u_i(x_i) = M_i > \alpha$ . This follows  $x_i \in L_i^\alpha$  and  $(x_i, d_i) \in L_i^\alpha \times \mathbb{R}_+$ .

So,  $\widehat{L}_i^\alpha \subset L_i^\alpha \times \mathbb{R}_+$ . It is obvious  $L_i^\alpha \times \mathbb{R}_+ \subset \widehat{L}_i^\alpha$ .

**Case 2:**  $\alpha \geq M_i$ . We claim that  $\widehat{L}_i^\alpha = S_i \times \left\{d_i : d_i \geq \frac{\alpha - M_i}{\mu}\right\}$ . Indeed, if  $\widehat{u}_i(x_i, d_i) \geq \alpha$ , then  $x_i \in S_i$ . In this case,  $\widehat{u}_i(x_i, d_i) = M_i + \mu d_i \geq \alpha$ , and hence  $d_i \geq \frac{\alpha - M_i}{\mu}$ . The converse is obvious.



It is also obvious that  $S_i$  is closed and convex. We have proved that  $\hat{u}_i$  is upper semicontinuous and quasi-concave for every  $i$ .

We now prove that  $\hat{u}_i$  is strictly quasi-concave.

Indeed, take  $M_i = u_i(x)$  with  $x \in S_i$  and  $(x_i, d_i), (x'_i, d'_i) \in X_i \times R_+$  such that  $\hat{u}_i(x'_i, d'_i) > \hat{u}_i(x_i, d_i)$ . For any  $\lambda \in ]0, 1[$ , we verify that

$$\hat{u}_i(\lambda x_i + (1 - \lambda)x'_i, \lambda d_i + (1 - \lambda)d'_i) > \hat{u}_i(x_i, d_i).$$

Since  $\hat{u}_i(x'_i, d'_i) > \hat{u}_i(x_i, d_i)$ , we can consider the following cases:

**Case 1:**  $x'_i \in S_i, x_i \in S_i$ . We have

$$\hat{u}_i(x_i, d_i) = M_i + \mu d_i, \hat{u}_i(x'_i, d'_i) = M_i + \mu d'_i.$$

It follows that  $d'_i > d_i$ . Hence

$$\lambda d_i + (1 - \lambda)d'_i > \lambda d_i + (1 - \lambda)d_i = d_i.$$

Since  $\lambda x_i + (1 - \lambda)x'_i \in S_i$ , we deduce

$$\begin{aligned} \hat{u}_i(\lambda x_i + (1 - \lambda)x'_i, \lambda d_i + (1 - \lambda)d'_i) &= \\ M_i + \mu(\lambda d_i + (1 - \lambda)d'_i) &> M_i + \mu d_i = \hat{u}_i(x_i, d_i). \end{aligned}$$

**Case 2:**  $x'_i \in S_i, x_i \notin S_i$ . It implies  $u_i(x'_i) > u_i(x_i)$ . Since  $u_i$  is a strictly quasi-concave function, we obtain

$$u_i(\lambda x_i + (1 - \lambda)x'_i) > u_i(x_i).$$

**2a:** If  $\lambda x_i + (1 - \lambda)x'_i \in S_i$ , then

$$\hat{u}_i(\lambda x_i + (1 - \lambda)x'_i, \lambda d_i + (1 - \lambda)d'_i) = M_i + \mu(\lambda d_i + (1 - \lambda)d'_i) > u_i(x_i) = \hat{u}_i(x_i, d_i).$$

**2b:** If  $\lambda x_i + (1 - \lambda)x'_i \notin S_i$ , then

$$\hat{u}_i(\lambda x_i + (1 - \lambda)x'_i, \lambda d_i + (1 - \lambda)d'_i) = u_i(\lambda x_i + (1 - \lambda)x'_i) > u_i(x_i) = \hat{u}_i(x_i, d_i).$$

**Case 3**  $x'_i \notin S_i, x_i \notin S_i$ . We have

$$\hat{u}_i(x_i, d_i) = u_i(x_i), \hat{u}_i(x'_i, d'_i) = u_i(x'_i).$$

This follows  $u_i(x'_i) > u_i(x_i)$ . Similarly as above we consider

**3a:** If  $\lambda x_i + (1 - \lambda)x'_i \in S_i$ , then

$$\begin{aligned}\hat{u}_i(\lambda x_i + (1 - \lambda)x'_i, \lambda d_i + (1 - \lambda)d'_i) &= M_i + \mu(\lambda d_i + (1 - \lambda)d'_i) \\ &> u_i(x_i) = \hat{u}_i(x_i, d_i).\end{aligned}$$

**3b:** If  $\lambda x_i + (1 - \lambda)x'_i \notin S_i$ , then

$$\hat{u}_i(\lambda x_i + (1 - \lambda)x'_i, \lambda d_i + (1 - \lambda)d'_i) = u_i(\lambda x_i + (1 - \lambda)x'_i) > u_i(x_i) = \hat{u}_i(x_i, d_i).$$

We have proved that the function  $\hat{u}_i$  is strictly quasi-concave.

It remains to prove that the  $\hat{u}_i$  has no satiation point.

Indeed, let  $(x_i, d_i) \in X_i \times \mathbb{R}_+$ . We consider the following cases

**Case 1:**  $x_i \notin S_i$ . Take  $x'_i \in X_i$  such that  $u_i(x'_i) > u_i(x_i)$  and  $d'_i = d_i$ . We have  $\hat{u}_i(x'_i, d_i) \geq u_i(x'_i) > u_i(x_i) = \hat{u}_i(x_i, d_i)$ .

**Case 2:**  $x_i \in S_i$ . Take  $x'_i = x_i$  and  $d'_i > d_i$ . We have

$$\hat{u}_i(x'_i, d'_i) = \hat{u}_i(x'_i) + \mu d'_i > u_i(x_i) + \mu d_i = \hat{u}_i(x_i, d_i).$$

We have proved that the  $\hat{u}_i$  has no satiation point.

Let us consider the feasible set  $\hat{A}$  of  $\hat{\mathcal{E}}$ . We have:

$$\hat{A} = \{((x_i, d_i)_{i \in I}, (y_j, 0)_{j \in J}) : \forall i, x_i \in X_i, d_i \in \mathbb{R}_+, \forall j, y_j \in Y_j \text{ and}$$

$$\sum_{i \in I} x_i = \sum_{i \in I} e_i + \sum_{j \in J} y_j, \sum_{i \in I} d_i = \sum_{i \in I} \delta_i.\}$$

It is obvious that  $\hat{A}$  is compact.

It is also obvious that Assumptions  $(H_1)$ ,  $(H_2)$ ,  $(H_3)$  are fulfilled in economy  $\hat{\mathcal{E}}$ .

Apply Theorem 2, part (i).

There exists a quasi-equilibrium  $((x_i^*, d_i^*)_{i \in I}, (y_j^*, 0)_{j \in J}, (p^*, q^*))$  with  $(p^*, q^*) \neq (0, 0)$ . It satisfies:

$$(i) \sum_{i \in I} (x_i^*, d_i^*) = \sum_{i \in I} (e_i, \delta_i) + \sum_{j \in J} (y_j^*, 0),$$

$$(ii) \text{ for any } i \in I, p^* \cdot x_i^* + q^* d_i^* = p^* \cdot e_i + \sum_{j \in J} \theta_{ij} \sup(p^* \cdot Y_j + q^* \times 0) + q^* \delta_i,$$

and

$$(iii) \text{ for any } j \in J, p^* \cdot y_j^* = \sup(p^* \cdot Y_j).$$

Observe that since  $\mu > 0$ , the price  $q^*$  must be nonnegative.

We claim that  $((x_i^*)_{i \in I}, (y_j^*)_{j \in J}, p^*)$  is an equilibrium with dividends  $(q^* \delta_i)_{i \in I}$ . Indeed, first, we have

$$\forall i \in I, p^* \cdot x_i^* \leq p^* \cdot e_i + \sum_{j \in J} \theta_{ij} p^* \cdot y_j^* + q^* \delta_i.$$

Now, let  $x_i \in X_i, u_i(x_i) > u_i(x_i^*)$ . That implies  $x_i^* \notin S_i$  and hence  $\hat{u}_i(x_i^*, d_i^*) = u_i(x_i^*)$ . We also have  $\hat{u}_i(x_i, 0) = u_i(x_i)$ . That means  $\hat{u}_i(x_i, 0) > \hat{u}_i(x_i^*, d_i^*)$ . This implies

$$p^* \cdot x_i = p^* \cdot x_i + q^* \times 0 \geq p^* \cdot e_i + \sum_{j \in J} \theta_{ij} \sup p^* \cdot Y_j + (q^* \delta_i).$$

We claim that

$$p^* \cdot x_i > p^* \cdot e_i + \sum_{j \in J} \theta_{ij} \sup p^* \cdot Y_j + (q^* \delta_i).$$

Assume the contrary, i.e.

$$p^* \cdot x_i = p^* \cdot e_i + \sum_{j \in J} \theta_{ij} \sup p^* \cdot Y_j + (q^* \delta_i). \quad (1)$$

Then, since

$$e_i \in \text{int}(X_i - \sum_{j \in J} \theta_{ij} Y_j),$$

we have

$$\inf p^* \cdot (X_i - \sum_{j \in J} \theta_{ij} Y_j) < p^* \cdot e_i.$$

This means that there exists  $x'_i \in X_i, y'_j \in Y_j$  such that

$$p^* \cdot (x'_i - \sum_{j \in J} \theta_{ij} y'_j) < p^* \cdot e_i$$

which implies

$$p^* \cdot x'_i < \sum_{j \in J} \theta_{ij} p^* \cdot y'_j + p^* \cdot e_i \leq \sum_{j \in J} \theta_{ij} p^* \cdot y_j^* + p^* \cdot e_i + q^* \delta_i. \quad (2)$$

Let  $x_i^\lambda = \lambda x'_i + (1 - \lambda)x_i$  with  $\lambda > 0$ . Since  $\{x_i : u_i(x_i) > u_i(x_i^*)\}$ , by assumption, is relatively open, we have

$$u_i(x_i^\lambda) > u_i(x_i^*). \quad (3)$$

for every  $\lambda$  sufficiently small. On the other hand, from (1) and (2) we have

$$\begin{aligned} p^* \cdot (\lambda x'_i + (1 - \lambda)x_i) &= \lambda p^* \cdot x'_i + (1 - \lambda)p^* \cdot x_i \\ &< \lambda \left( \sum_{j \in J} \theta_{ij} p^* \cdot y_j^* + p^* \cdot e_i + q^* \delta_i \right) + (1 - \lambda) \left( \sum_{j \in J} \theta_{ij} p^* \cdot y_j^* + p^* \cdot e_i + q^* \delta_i \right) \end{aligned}$$

or

$$p^* \cdot x_i^\lambda + q^* \times 0 < p^* \cdot e_i + \sum_{j \in J} \theta_{ij} p^* \cdot y_j^* + q^* \delta_i. \quad (4)$$

Since  $\hat{u}_i(x_i^\lambda, 0) = u_i(x_i^\lambda)$  and  $\hat{u}_i(x_i^*, d_i^*) = u_i(x_i^*)$ , relations (3) and (4) contradict the fact that  $((x_i^*, d_i^*)_{i \in I}, (y_j^*)_{j \in J}, (p^*, q^*))$  is a quasi-equilibrium of the intermediary economy. ■

**Corollary 4** *Assume  $(H_1) - (H_4)$ . Let  $((x_i^*)_{i \in I}, (y_j^*)_{j \in J}, p^*)$  be an equilibrium with dividends  $(d_i^*)$ . If consumer  $i$  is non-satiated, then*

$$p^* \cdot x_i^* = p^* \cdot e_i + \sum_{j \in J} \theta_{ij} \cdot \sup p^* \cdot Y_j + q^* \delta_i,$$

and  $p^* \neq 0$ .

*Suppose that every consumer is non-satiated. Then an equilibrium with dividends will be reduced to a Walras equilibrium. That is the dividend is zero and the equilibrium price is non-zero.*

**Proof.** First, we prove that, if  $x_i^*$  is not a satiation point, then  $q^* d_i^* = 0$ . Indeed, let  $u_i(x_i) = \hat{u}_i(x_i, 0) > u_i(x_i^*) = \hat{u}_i(x_i^*, d_i^*)$ . We then have

$$p^* \cdot x_i \geq p^* \cdot e_i + \sum_{j \in J} \theta_{ij} \sup p^* \cdot Y_j + q^* \delta_i = p^* \cdot x_i^* + q^* d_i^*.$$

For any  $\lambda \in ]0, 1[$ , from the strict quasi-concavity of  $u_i$ , we have  $u_i(\lambda x_i + (1 - \lambda)x_i^*) > u_i(x_i^*)$  and hence  $p^* \cdot (\lambda x_i + (1 - \lambda)x_i^*) \geq p^* \cdot x_i^* + q^* d_i^*$ . Letting  $\lambda$  converge to zero, we obtain  $q^* d_i^* \leq 0$ . Thus  $q^* d_i^* = 0$ . That means that a consumer who does not meet satiation point will sell her/his endowment

of the additional good if  $q^* > 0$ . Observe also that  $p^* \neq 0$  (if not we have  $0 = q^* \delta_i$ ; this implies  $q^* = 0$  : a contradiction with  $(p^*, q^*) \neq 0$ ).

One deduces from that, if  $x_i^*$  is not a satiation point for every  $i \in I$ , then  $q^* = 0$ , since  $\sum_{i \in I} d_i = \sum_{i \in I} \delta_i > 0$ . In this case,  $p^* \neq 0$ , and  $((x_i^*)_{i \in I}, (y_j^*)_{j \in J}, p^*)$  is a Walras equilibrium. ■

**Remark 5** (1) We can replace  $(H_3)$  by  $(H_3 \text{ bis})$ : "The total production set  $\sum_{j \in J} Y_j$  is closed, non-empty and convex" as in Florig and Yildiz (2002), i.e., we do not require every  $Y_j$  be convex. Indeed, we replace the sets  $Y_j$  by their closed convex hulls  $\overline{co}Y_j$ . Let  $((x_i^*), (y_j^*), p^*)$  be an equilibrium with dividends  $(d_i^*)$  of this new economy. This implies that every  $y_j^*$  is in  $\overline{co}Y_j$ . It is obvious that for any  $j$

$$p^* \cdot y_j^* = \max_{y \in \overline{co}Y_j} p^* \cdot y = \sup_{y \in Y_j} p^* \cdot y.$$

By assumption,  $\sum_j Y_j$  is closed and convex. We then have  $\sum_j Y_j = \sum_j \overline{co}Y_j$ <sup>1</sup>. Hence there exist  $(\zeta_j^*) \in \Pi_j Y_j$  such that  $\sum_j \zeta_j^* = \sum_j y_j^*$ . Since  $\sum_i x_i^* = \sum_i e_i + \sum_j y_j^*$ , and since  $p^* \cdot \zeta_j^* \leq p^* \cdot y_j^*, \forall j$ , we must have  $p^* \cdot \zeta_j^* = p^* \cdot y_j^* = \max p^* \cdot Y_j$  for every  $j$ . That means that  $((x_i^*), (\zeta_j^*), p^*)$  is an equilibrium with dividends for the initial economy.

(2) Let  $I_1 = \{i \in I : x_i^* \text{ is not a satiation point}\}$ , and  $I_2 = I \setminus I_1$ . From Corollary 4,  $q^* d_i^* = 0$ , for any  $i \in I_1$ . Thus  $\sum_{i \in I_1} q^* \delta_i = \sum_{i \in I_2} q^* d_i^* - \sum_{i \in I_2} q^* \delta_i$ . This shows that the group of agents who meet satiation buy the additional good from the group of agents who do not meet satiation.

## 4 No-arbitrage condition and existence of equilibrium with dividends

If we assume that  $0 \in Y_j$  for every  $j$ , and if  $((x_i^*)_{i \in I}, (y_j^*)_{j \in J}, p^*)$  is an equilibrium with dividends, we will have

$$p^* \cdot e_i = p^* \cdot e_i + \sum_j \theta_{ij} p^* \cdot 0 \leq p^* \cdot e_i + \sum_{j \in J} \theta_{ij} p^* \cdot y_j^* + q^* \delta_i.$$

---

<sup>1</sup>It comes from three facts. (i) We always have  $\sum_j \overline{co}Y_j = \overline{co} \sum_j Y_j$  (see e.g. Florenzano, Le Van and Gourdel, 2001, p. 16), (ii)  $\sum_j \overline{co}Y_j \subset \overline{\sum_j \overline{co}Y_j}$  and  $\sum_j Y_j$  is closed and convex.

Hence, for every  $i$ , we have  $u_i(x_i^*) \geq u_i(e_i)$ . We therefore define the set of *individually rational feasible allocations*  $\tilde{A}$ . More precisely:

$$\tilde{A} = \left\{ ((x_i), (y_j)) \in \prod_{i \in I} X_i \times \prod_{j \in J} Y_j : \sum_{i \in I} x_i = \sum_{i \in I} e_i + \sum_{j \in J} y_j, \forall i, u_i(x_i) \geq u_i(e_i) \right\}.$$

We will replace  $(H_4)$  by

$(H_4bis)$  The set  $\tilde{A}$  is compact.

We have the following result:

**Theorem 6** (i) Assume  $(H_1), (H_2), (H_3), (H_4bis), (H_5)$ , for every  $j$ ,  $0 \in Y_j$  and

$$\forall i, \forall x_i \in X_i, \exists x'_i \in X_i \text{ such that } u_i(x'_i) > u_i(x_i).$$

Then there exists a Walras equilibrium.

(ii) Assume  $(H_1), (H_2), (H_3), (H_4bis), (H_5)$  and for every  $j$ ,  $0 \in Y_j$ . Then there exists an equilibrium with dividends.

**Proof.** The proof is similar to the one of Theorem 2. One just replaces the feasible set  $A$  by the set of individually rational feasible allocations  $\tilde{A}$ . ■

Let  $P_i = \{x_i \in X_i : u_i(x_i) \geq u_i(e_i)\}$ , and  $W_i$  be the recession cone of  $P_i$ . Elements in  $W_i$  which are different from zero will be called useful vectors for agent  $i$  (see Werner, 1987). Let  $Z_j$  denote the recession cone of  $Y_j$ . Take some  $\gamma_j \in Y_j$ . Then  $\gamma_j + \lambda z_j \in Y_j, \forall \lambda \geq 0, \forall z_j \in Z_j$ . We call useful production vector for firm  $j$  any vector  $z_j \in Z_j \setminus \{0\}$  (the producer can produce an infinitely large quantity  $\gamma_j + \lambda z_j, \lambda \geq 0$ ).

Let  $p \in \mathbb{R}^l$ . We say that there exists an *opportunity of arbitrage* associated with  $p$  if either there exists  $i \in I$ ,  $w_i \in W_i \setminus \{0\}$ , such that  $p \cdot w_i \leq 0$ , or there exists  $j \in J$ ,  $z_j \in Z_j$ , such that  $p \cdot z_j > 0$ . In other words, with such a price  $p$ , either the consumer  $i$  will increase without bounds her/his consumption or firm  $j$  will produce an infinite quantity.

A price vector  $p \in \mathbb{R}^l$  is a *no-arbitrage price* for the economy if  $\forall i \in I$ ,  $w_i \in W_i \setminus \{0\} \implies p \cdot w_i > 0$ , and  $\forall j \in J$ ,  $z_j \in Z_j \implies p \cdot z_j \leq 0$ .

We introduce the following **No-Arbitrage Condition**:

(NA) There exists a no-arbitrage price for the economy.

**Remark 7** Our No-Arbitrage Condition coincides with the one for an exchange economy, i.e. when  $Y_j = \{0\}, \forall j$ .

Let us replace  $(H_3)$  by

$(H_{3ter})$  For each  $j \in J$ , the set  $Y_j$  is nonempty closed convex and  $Y = \sum_{j \in J} Y_j$  is closed. Moreover, for every  $j$ ,  $0 \in Y_j$  and  $Y \cap -Y = \{0\}$ .

We have the following result

**Theorem 8** (i) Assume  $(H_1), (H_2), (H_{3ter}), (H_5)$  and  $(NA)$ . Then there exists an equilibrium with dividends.

(ii) Assume the following No-Halfline Condition :

$(NHL)$  For  $i \in I$ , if  $w_i \in W_i \setminus \{0\}$ , then for any  $x \in P_i$ , there exists  $\lambda > 0$ , such that  $u_i(x + \lambda w_i) > u_i(x)$ .

Then:

$((x_i^*)_{i \in I}, (y_j^*)_{j \in J}, p^*)$  is an equilibrium with dividends  $\Rightarrow p^*$  is a no-arbitrage price.

**Proof.** (i) It suffices to prove that  $\tilde{A}$  is compact. Assume the contrary. Then there is a sequence  $\left( (x_i^n)_i, (y_j^n)_j \right)_{n=1, \dots, \infty} \in \tilde{A}$  such that  $\sigma_n = \sum_i \|x_i^n\| + \sum_j \|y_j^n\| \rightarrow +\infty$  when  $n \rightarrow \infty$ . Since

$$\frac{\sum_i x_i^n}{\sigma_n} = \frac{\sum_i e_i}{\sigma_n} + \frac{\sum_j y_j^n}{\sigma_n}$$

We can assume, without loss of generality, that

$$\left( \left( \frac{x_i^n}{\sigma_n} \right)_i, \left( \frac{y_j^n}{\sigma_n} \right)_j \right) \rightarrow \left( (w_i)_i, (z_j)_j \right) \in \left( \left( \prod_i W_i \right) \times \left( \prod_j Z_j \right) \right) \setminus \{0\}.$$

Moreover, we have

$$\sum_i w_i = \sum_j z_j.$$

Let  $p$  be a no-arbitrage price. If  $(w_i)_i \neq 0$ , we have a contradiction:  $0 < p \cdot \sum_i w_i = p \cdot \sum_j z_j \leq 0$ . If  $(w_i)_i = 0$ , then  $\sum_j z_j = 0$ . We have:  $\sum_{k \neq j} z_k = -z_j$ . From  $(H_{3ter})$ ,  $\sum_{k \neq j} z_k \in Y$  and  $z_j \in Y$ . Hence  $z_j \in Y \cap -Y$ . This implies  $z_j = 0$ . We have shown that, in this case, we have  $(z_j)_j = 0$  and a contradiction with  $\left( (w_i)_i, (z_j)_j \right) \neq 0$ .

We have proved that  $\tilde{A}$  is compact.

(ii) Let  $((x_i^*)_{i \in I}, (y_j^*)_{j \in J}, p^*)$  be an equilibrium with dividends. It is obvious that  $p^* \cdot z_j \leq 0$ , for every  $z_j \in Z_j$  since  $y_j^* + z_j \in Y_j$  and  $p^* \cdot y_j^* = \max p^* \cdot Y_j$ .

We have two cases.

**Case 1.** There exists some  $i \in I$  such that  $x_i^*$  is not a satiation point. From Corollary 4,  $p^* \neq 0$ . If  $w_i \in W_i \setminus \{0\}$ , then Condition  $(NHL)$  implies  $u_i(x_i^* + \lambda w_i) > u_i(x_i^*)$ , for some  $\lambda > 0$ . Since  $((x_i^*)_{i \in I}, (y_j^*)_{j \in J}, p^*)$  is an equilibrium, we have  $p^*.w_i > 0$ .

**Case 2.** For any  $i \in I$ ,  $x_i^*$  is a satiation point. Condition  $(NHL)$  implies that  $W_i = \{0\}$ , for every  $i$ . No-arbitrage Condition is satisfied in this case with  $p^*$ . ■

**Remark 9** *The No Halfline Condition is satisfied with strictly concave functions.*

## 5 Appendix 1: Proof of Theorem 2

### 5.1 Gale-Nikaido-Debreu Lemma

We will make use of the following lemma the proof of which can be found in Florenzano and Le Van (1986):

**Lemma 10 (Gale-Nikaido-Debreu)** *Let  $P$  be a closed nonempty convex cone in the linear space  $\mathbb{R}^l$ . Let  $P^0$  be the polar cone of  $P$  and  $S$  be the unit sphere in  $\mathbb{R}^l$ . Suppose that the multivalued mapping  $Z$  from  $S \cap P$  to  $\mathbb{R}^l$  is upper semicontinuous and  $Z(p)$  is nonempty convex compact. Suppose further that for every  $p \in S \cap P$ ,  $\exists z \in Z(p)$  such that  $p.z \leq 0$ . Then there exists  $\bar{p} \in S \cap P$  satisfying*

$$Z(\bar{p}) \cap P^0 \neq \emptyset,$$

where  $P^0 = \{q \in \mathbb{R}^l : q.p \leq 0, \forall p \in P\}$ .



## 5.2 Proof of Theorem 2

We consider a sequence of truncated economies.

Let  $B(0, n)$  denote the ball centered at 0 with radius  $n$ . Let

$$X_i^n = X_i \cap B(0, n) \quad , \quad Y_j^n = Y_j \cap B(0, n)$$

where  $i \in I, j \in J$ . Since  $e_i \in X_i$ , we have  $e_i \in X_i^n$  for all  $n$  is large enough. For every  $(p, q) \in S \cap (\mathbb{R}^l \times \mathbb{R}_+)$ , where  $S$  is the unit sphere of  $\mathbb{R}^{l+1}$ , define the multivalued mapping

$$\xi_i^n, Q_i^n : \mathbb{R}^l \times \mathbb{R}_+ \longrightarrow X_i$$

by setting

$$\begin{aligned} \xi_i^n(p, q) &= \left\{ x_i \in X_i^n : p \cdot x_i \leq p \cdot e_i + \sum_{j \in J} \theta_{ij} \Pi_j^n(p) + q \right\}, \\ Q_i^n(p, q) &= \left\{ x_i \in \xi_i^n(p, q) : \text{if } x'_i \in X_i^n \text{ with } u_i(x'_i) > u_i(x_i) \text{ then } p \cdot x'_i \geq \right. \\ &\quad \left. p \cdot e_i + \sum_{j \in J} \theta_{ij} \Pi_j^n(p) + q \right\}, \text{ where } \Pi_j^n(p) = \max p \cdot Y_j^n. \end{aligned}$$

Under the assumptions mentioned in Theorem 2 we have the following lemma:

**Lemma 11** *For each  $i \in I$  the mapping  $Q_i^n$  is upper semicontinuous having nonempty compact convex values.*

**Proof.** From the definition it is easy to see that  $\xi_i^n$  is upper semicontinuous having nonempty convex compact values. From the definition of the mapping  $Q_i^n$  we have the following properties:

Let  $x \in \xi_i^n(p, q)$  and  $u_i(x) = \max u_i(x_i)$ , with  $x_i \in \xi_i^n(p, q)$  then  $x \in Q_i^n(p, q)$ . Indeed, let  $x'_i \in X_i^n$  and  $u_i(x'_i) > u_i(x)$ , then  $x'_i \notin \xi_i^n(p, q)$ . Hence by the definition of this set we have  $p \cdot x'_i \geq p \cdot e_i + \sum_{j \in J} \theta_{ij} \Pi_j^n(p) + q$ , and therefore  $x \in Q_i^n(p, q)$ . This implies that  $Q_i^n(p, q)$  nonempty for every  $(p, q) \in S \cap (\mathbb{R}^l \times \mathbb{R}_+)$ . For every  $x_i, y_i \in Q_i^n(p, q)$  and  $\lambda \in [0, 1]$ , since  $\xi_i^n(p, q)$  is convex we have  $\lambda x_i + (1 - \lambda)y_i \in \xi_i^n(p, q)$ . On the other hand, since  $u_i$  is strictly quasiconcave,  $u_i(\lambda x_i + (1 - \lambda)y_i) > \min(u_i(x_i), u_i(y_i))$ . Hence, for each  $x'_i \in X_i^n$  and  $u_i(x'_i) > u_i(\lambda x_i + (1 - \lambda)y_i)$ , it follows that  $u_i(x'_i) > \min(u_i(x_i), u_i(y_i))$ . Thus  $p \cdot x'_i \geq p \cdot e_i + \sum_{j \in J} \theta_{ij} \Pi_j^n(p) + q$ . Hence  $\lambda x_i + (1 - \lambda)y_i \in Q_i^n(p, q)$  which means that  $Q_i^n(p, q)$  is convex.

The mapping  $Q_i^n$  is closed. Indeed, let

$$(p^k, q^k, x_i^k) \in \text{graph } Q_i^n$$

and assume that  $(p^k, q^k) \rightarrow (p, q), x_i^k \rightarrow x_i$ . Since  $x_i^k \in Q_i^n(p^k, q^k) \subset \xi_i^n(p^k, q^k)$  and  $\xi_i^n$  is closed, we have  $x_i \in \xi_i^n(p, q)$ . On the other hand, let  $x'_i \in X_i^n$  with  $u_i(x'_i) > u_i(x_i)$ , by the upper semicontinuity of  $u_i$  we see that  $u_i(x'_i) > u_i(x_i^k)$  for all  $k$  large enough. Since  $x_i^k \in Q_i^n(p^k, q^k)$ , we have

$$p^k \cdot x'_i \geq p^k \cdot e_i + \sum_{j \in J} \theta_{ij} \prod_j^n(p^k) + q^k.$$

Letting  $k \rightarrow +\infty$  we obtain

$$p \cdot x'_i \geq p \cdot e_i + \sum_{j \in J} \theta_{ij} \prod_j^n(p) + q.$$

This implies that  $x_i \in Q_i^n(p, q)$ . Hence  $Q_i^n$  is closed. But, since

$$Q_i^n(p, q) \subset \xi_i^n(p, q) \subset X_i^n$$

for all  $(p, q) \in S \cap (\mathbb{R}^l \times \mathbb{R}_+)$ ,  $n \geq 1$  and  $X_i^n$  is compact, we see that  $Q_i^n$  is a compact mapping. Hence  $Q_i^n$  is upper semicontinuous. ■

a) Under assumptions  $(H_1) - (H_4)$  we now show that there exists quasi-equilibrium. Let  $\Phi_j^n(p)$  denote the solution-set of  $\prod_j^n(p)$ , that means  $y_j \in \Phi_j^n(p)$  if and only if  $p \cdot y_j = \max p \cdot Y_j^n$ . Define the mapping  $z^n$  by setting, for each  $(p, q) \in S \cap (\mathbb{R}^l \times \mathbb{R}_+)$ ,

$$z^n(p, q) = \left( \sum_{i \in I} Q_i^n(p, q) - \sum_{i \in I} e_i - \sum_{j \in J} \phi_j^n(p) \right) \times \{-|I|\}$$

where  $S$  stands for the unit sphere in  $\mathbb{R}^{l+1}$ . By virtue of Lemma 11, from the assumptions of the theorem it is easy to see that  $z^n$  is upper semicontinuous having nonempty convex compact values. Note that for any  $x$  in  $z^n(p, q)$  we can write

$$x = \left( \sum_{i \in I} x_i^n - \sum_{i \in I} e_i - \sum_{j \in J} y_j^n \right) \times (-|I|)$$

where  $x_i^n \in Q_i^n(p, q)$  and  $y_j^n \in \Phi_j^n(p)$ . Since  $x_i^n \in Q_i^n(p, q)$ , that implies

$$p \cdot x_i^n \leq p \cdot e_i + \sum_{j \in J} \theta_{ij} \prod_j^n(p) + q = p \cdot e_i + \sum_{j \in J} \theta_{ij} p \cdot y_j^n + q$$

or

$$p \cdot \sum_{i \in I} x_i^n \leq p \cdot \sum_{i \in I} e_i + \sum_{i \in I} \sum_{j \in J} \theta_{ij} p \cdot y_j^n + |I|q = p \cdot \sum_{i \in I} e_i + p \cdot \sum_{j \in J} y_j^n + |I|q.$$

Thus

$$p.(\sum_{i \in I} x_i^n - \sum_{i \in I} e_i - \sum_{j \in J} y_j^n) - |I|q \leq 0.$$

Hence  $(p, q).x \leq 0$  for every  $(p, q) \in S \cap \mathbb{R}^l \times \mathbb{R}_+$ , and  $x \in z^n(p, q)$ . Applying the Gale-Nikaido-Debreu Lemma, we can conclude that there exists  $(p^n, q^n) \in S \cap (\mathbb{R}^l \times \mathbb{R}_+)$  such that

$$z^n(p^n, q^n) \cap (\mathbb{R}^l \times \mathbb{R}_+)^0 \neq \emptyset.$$

Since  $(\mathbb{R}^l \times \mathbb{R}_+)^0 = (O_{\mathbb{R}^l} \times \mathbb{R}_-)$ , it follows that for every  $i \in I, j \in J$  there exists  $x_i^n \in Q_i^n(p^n, q^n), y_j^n \in \Phi_j^n(p^n)$  satisfying

$$\sum_{i \in I} x_i^n - \sum_{i \in I} e_i - \sum_{j \in J} y_j^n = 0, \quad (5)$$

$$p^n.x_i^n \leq p^n.e_i + \sum_{j \in J} \theta_{ij} \prod_j^n(p^n) + q^n$$

for every  $i \in I$ , and

$$p^n.x'_i \geq p^n.e_i + \sum_{j \in J} \theta_{ij} \prod_j^n(p^n) + q^n. \quad (6)$$

for every  $x'_i \in X_i^n$  which satisfies  $u_i(x'_i) > u_i(x_i^n)$ .

From (5) we have  $(x_i^n, y_j^n) \in A$ . Since  $A$  is compact, without loss of generality, we may assume that

$$(x_i^n, y_j^n) \longrightarrow (x_i^*, y_j^*).$$

Since  $(p^n, q^n) \in S \cap (\mathbb{R}^l \times \mathbb{R}_+)$  and  $S \cap (\mathbb{R}^l \times \mathbb{R}_+)$  is compact, we can also assume  $(p^n, q^n) \longrightarrow (p^*, q^*)$ . From (5) and (6) it implies

$$\sum_{i \in I} x_i^* - \sum_{i \in I} e_i - \sum_{j \in J} y_j^* = 0, \quad (7)$$

$$p^*.x_i^* \leq p^*.e_i + \sum_{j \in J} \theta_{ij} \prod_j(p^*) + q^* \quad \text{for every } i \in I, \quad (8)$$

where  $\prod_j(p^*) = \max\{p^*.Y_j\}$ .

Let  $x_i \in X_i$  satisfy  $u_i(x_i) > u_i(x^*)$ . Define

$$x_i^\lambda = \lambda x_i + (1 - \lambda)x_i^*,$$

where  $\lambda \in (0, 1]$ . Since  $u_i$  is strictly quasiconcave, it implies  $u_i(x_i^\lambda) > u_i(x_i^*)$ . Moreover, since  $u_i$  is upper semicontinuous and  $x_i^n \rightarrow x_i^*$ , for every  $n$  large enough, we have  $u_i(x_i^\lambda) > u_i(x_i^n)$ . Thus by (6) we obtain

$$p^n \cdot x_i^\lambda \geq p^n \cdot e_i + \sum_{j \in J} \theta_{ij} \prod_j^n(p^n) + q^n$$

or

$$p^n \cdot (\lambda x_i + (1 - \lambda)x_i^*) \geq p^n \cdot e_i + \sum_{j \in J} \theta_{ij} \prod_j^n(p^n) + q^n.$$

Let  $n \rightarrow +\infty$  we obtain

$$\lambda p^* \cdot x_i + (1 - \lambda)p^* \cdot x_i^* \geq p^* \cdot e_i + \sum_{j \in J} \theta_{ij} \prod_j(p^*) + q^*.$$

Let  $\lambda \rightarrow 0$  we get

$$p^* \cdot x_i^* \geq p^* \cdot e_i + \sum_{j \in J} \theta_{ij} \prod_j(p^*) + q^*. \quad (9)$$

Then from (8) and (9) follows

$$p^* \cdot x_i^* = p^* \cdot e_i + \sum_{j \in J} \theta_{ij} \prod_j(p^*) + q^* \text{ for every } i \in I,$$

and hence

$$p^* \cdot \sum_{i \in I} x_i^* = p^* \cdot \sum_{i \in I} e_i + \sum_{i \in I} \sum_{j \in J} \theta_{ij} \prod_j(p^*) + |I|q^*.$$

or

$$p^* \cdot \left( \sum_{i \in I} x_i^* - \sum_{i \in I} e_i - \sum_{j \in J} y_j^* \right) = |I|q^*$$

But, from  $\sum_{i \in I} x_i^* - \sum_{i \in I} e_i - \sum_{j \in J} y_j^* = 0$  follows  $|I|q^* = 0$ . Hence  $q^* = 0$  and  $p^* \neq 0$ . Thus  $((x_i^*)_{i \in I}, (y_j^*)_{j \in J}, p^*)$  is a quasi-equilibrium.

b) Now we show that if, in addition,  $(H_5)$  is satisfied, then this quasi-equilibrium is in fact an equilibrium. Take  $x_i \in X_i$  such that  $u_i(x_i) > u_i(x_i^*)$ . By the just proved preceeding part we have

$$p^* \cdot x_i \geq p^* \cdot e_i + \sum_{j \in J} \theta_{ij} \sup p^* \cdot Y_j = p^* \cdot e_i + \sum_{j \in J} \theta_{ij} p^* \cdot y_j^*.$$

In contrary we suppose that

$$p^*.x_i = p^*.e_i + \sum_{j \in J} \theta_{ij} p^*.y_j^*. \quad (10)$$

Then, since

$$e_i \in \text{int}(X_i - \sum_{j \in J} \theta_{ij} Y_j),$$

we have

$$\inf p^*.(X_i - \sum_{j \in J} \theta_{ij} Y_j) < p^*.e_i.$$

This means that there exists  $x'_i \in X_i, y'_j \in Y_j$  such that

$$p^*.(x'_i - \sum_{j \in J} \theta_{ij} y'_j) < p^*.e_i$$

which implies

$$p^*.x'_i < \sum_{j \in J} \theta_{ij} p^*.y'_j + p^*.e_i \leq \sum_{j \in J} \theta_{ij} p^*.y_j^* + p^*.e_i. \quad (11)$$

Let  $x_i^\lambda = \lambda x'_i + (1 - \lambda)x_i$  with  $\lambda > 0$ . Since  $\{x_i : u_i(x_i) > u_i(x_i^*)\}$ , by assumption, is open, we have

$$u_i(x_i^\lambda) > u_i(x_i^*). \quad (12)$$

for every  $\lambda$  sufficiently small. On the other hand, from (10) and (11) we have

$$\begin{aligned} p^*.( \lambda x'_i + (1 - \lambda)x_i ) &= \lambda p^*.x'_i + (1 - \lambda)p^*.x_i \\ &< \lambda ( \sum_{j \in J} \theta_{ij} p^*.y_j^* + p^*.e_i ) + (1 - \lambda) ( \sum_{j \in J} \theta_{ij} p^*.y_j^* + p^*.e_i ) \end{aligned}$$

or

$$p^*.x_i^\lambda < p^*.e_i + \sum_{j \in J} \theta_{ij} p^*.y_j^*. \quad (13)$$

From (12) and (13) we arrive at a contradiction to the assumption that  $((x_i^*)_{i \in I}, (y_j^*)_{j \in J}, p^*)$  is a quasi-equilibrium. The theorem is proved.

## 6 Appendix 2: An example of economies where the no-arbitrage condition is satisfied

Consider a two-period economy with two consumers and one firm. There exists one consumption good, one capital good, two assets. In the second period, there are two states of nature. Firm produces in period 1. Consumer  $i$  consumes  $c_0^i$  in period 1,  $c_s^i$  in period 2 if state  $s$  occurs. She/he owns  $\alpha_i k_0$  capital stock ( $k_0$  is the initial capital stock,  $\alpha_i$  is the share between the two consumers of this capital stock). She/he buys in period 1,  $\theta_1^i$ ,  $\theta_2^i$  assets which yield in period 2,  $v_s^{i,1}\theta_1^i + v_s^{i,2}\theta_2^i$  consumption goods if state  $s$  occurs. The preference of consumer  $i$  is represented by a concave, increasing function  $u^i$ . Consumer  $i$  solves the problem ( $\mathcal{P}$ ):

$$\max u^i(c_0^i, c_1^i, c_2^i)$$

under the constraints

$$p_0 c_0^i + q \cdot \theta^i \leq \alpha_i r k_0 + \beta_i \pi^*$$

and

$$0 \leq c_s^i \leq e_s^i + v_s^{i,1}\theta_1^i + v_s^{i,2}\theta_2^i$$

where  $p_0$  is the price of consumption good in period 1,  $q$  is the price of assets,  $\pi^*$  is the profit of firm,  $\beta_i$  is the share of profit,  $r$  is the price of the capital good and  $e_s^i$  is the initial endowment in state  $s$ .

Firm solves the problem ( $\mathcal{Q}$ ):

$$\pi^* = \max_k \{p_0 F(k) - rk\}$$

where  $F$  is a concave production function, increasing and  $F(0) = 0$ .

An equilibrium is a list  $(p_0^*, q^*, r^*, c_0^{*i}, c_1^{*i}, c_2^{*i}, k^*)$  such that

(i)  $(c_0^{*i}, c_1^{*i}, c_2^{*i})$  solve problem ( $\mathcal{P}$ ) with  $p_0 = p_0^*, q = q^*, r = r^*$ ,

(ii)  $k^*$  solves ( $\mathcal{Q}$ ) with  $p_0 = p_0^*, r = r^*$ ,

and (iii)

$$\begin{aligned} c_0^{*1} + c_0^{*2} &= F(k^*) \\ c_s^{*i} &= e_s^i + v_s^{i,1}\theta_1^{*i} + v_s^{i,2}\theta_2^{*i}, \quad \forall s = 1, 2 \\ \sum_{i=1}^2 \theta_1^{*i} &= 0, \quad \sum_{i=1}^2 \theta_2^{*i} = 0 \end{aligned}$$

and finally

$$k^* = k_0.$$

Since the functions  $u^i$  are increasing, the equilibrium problem is equivalent to the following.

$(c_0^{*i}, \theta_1^{*i}, \theta_2^{*i})$  solve:

$$\max u^i(c_0^i, e_1^i + v_1^{i,1}\theta_1^i + v_1^{i,2}\theta_2^i, e_2^i + v_2^{i,1}\theta_1^i + v_2^{i,2}\theta_2^i)$$

under the constraints

$$p_0^* c_0^i + q^* \cdot \theta^i \leq \alpha_i r^* k_0 + \beta_i \pi^*$$

where  $\pi^* = \max_k p_0^* F(k) - r^* k = p_0^* F(k^*) - r^* k^*$  and

$$c_0^{*1} + c_0^{*2} = F(k^*),$$

$$\sum_{i=1}^2 \theta_1^{*i} = 0, \sum_{i=1}^2 \theta_2^{*i} = 0$$

$$k^* = k_0.$$

Let  $A_i = \{(\theta_1, \theta_2) : e_s^i + v_s^{i,1}\theta_1 + v_s^{i,2}\theta_2 \geq 0, \text{ for } s = 1, 2\}$ .

The consumption set for consumer  $i$  is  $X_i = \mathbb{R}_+ \times \mathbb{R}_+ \times A_i$  (the second factor corresponds to the capital good). Let  $O^+ A_i$  denote the recession cone of  $A_i$ .

Then the recession cone of  $X_i$  is  $W_i = \mathbb{R}_+ \times \mathbb{R}_+ \times O^+ A_i$ .

The production set for firm is

$$Y = \{(y, -k, \theta_1, \theta_2) \in \mathbb{R}_+ \times \mathbb{R}_- \times \{0\} \times \{0\} : y \leq F(k)\}.$$

Its recession cone is  $Z = \mathbb{R}_- \times \mathbb{R}_- \times \{0\} \times \{0\}$ .

Let  $S_i = \{(q_1, q_2) : q_1 w_1 + q_2 w_2 > 0, \forall (w_1, w_2) \in O^+ A_i \setminus \{(0, 0)\}\}$ .

Assume  $S = S_1 \cap S_2 \neq \emptyset$ . Then the No-arbitrage Condition holds. Indeed, let  $p = (1, 1, s)$  with  $s \in S$ . Then we have  $p \cdot w > 0$  for all  $w \in W_i \setminus \{0\}$  and  $p \cdot z < 0, \forall z \in Z \setminus \{0\}$ .

## References

- [1] Allouch, N., Le Van, C., Page Jr., F.H., 2002, The geometry of arbitrage and the existence of competitive equilibrium, Journal of mathematical economics, 38, No 4, 373-391.

- [2] Arrow, K.J., Debreu, G., 1954, Existence of equilibrium for a competitive economy, *Econometrica*, 22, N°3, 265-290.
- [3] Aumann, R.J., Drèze, J.H., 1986, Values of markets with satiation or fixed prices, *Econometrica*, 54, 1271-1318.
- [4] Cornet, B., Topuzu, M., Yildiz, A., 2003, Equilibrium theory with a measure space of possibly satiated consumers, *Journal of mathematical economics*, 39, No 3-4, 175-196.
- [5] Dana, R.A., Le Van, C., Magnien, F., 1997, General Equilibrium in Asset Markets With or Without Short-Selling, *Journal of mathematical analysis and applications*, 206, 567-588.
- [6] Dana, R.A., Le Van, C., Magnien, F., 1999, On the different notions of arbitrage and existence of equilibrium, *Journal of economic theory*, 87, 169-193.
- [7] Debreu, G., 1959, *Theory of value*, New York: Wiley.
- [8] Drèze, J.H., Muller, H., 1980, Optimality properties of rationing schemes, *Journal of economic theory*, 23, 131-149.
- [9] Florenzano, M. and Le Van, C., 1986, A note on Gale-Nikaido-Debreu lemma and the existence of general equilibrium, *Economics Letters*, 22, 107-110.
- [10] Florenzano, M., Le Van, C., in cooperation with Gourdel, P., 2001, *Finite dimensional convexity and optimization*, Springer.
- [11] Florig, M., Yildiz, A., 2002, Equilibrium without nonsatiation, *Cahiers de la MSE, série bleue*, 2002-107.
- [12] Kajii, A., 1996, How to discard non-satiation and free-disposal with paper money, *Journal of mathematical economics*, 25, 75-84.
- [13] Konovalov, A., 2005, The core of an economy with satiation, *Economic Theory*, 25, 711-719.
- [14] Makarov, V.L., 1981, Some results on general assumptions about the existence of economic equilibrium, *Journal of mathematical economics*, 8, 87-100.



- [15] Mas-Collel, A., 1992, Equilibrium theory with possibly satiated preferences, in Majumdar, M. (Ed.): Equilibrium and dynamics: proceedings of the essays in honour of David Gale, MacMillan, London, 201-213.
- [16] Nielsen, L.T., 1989, Asset market with short sellin, Review of economic studies, 56, 467-474.
- [17] Page Jr., 1987, On equilibrium in Hart's securities exchange model, Journal of economic theory, 41, 392-404.
- [18] Page Jr., F.H., Wooders, M., 1996, A necessary and sufficient condition for compactness of individually rational and feasible outcomes and existence of an equilibrium, Economics letters, 52, 153-162.
- [19] Werner, J., 1987, Arbitrage and the existence of competitive equilibrium, Econometrica, 55, 1403-1418.